

# Edge-connectivity and edge-disjoint spanning trees

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## Abstract

Given a graph  $G$ , for an integer  $c \in \{2, \dots, |V(G)|\}$ , define  $\lambda_c(G) = \min\{|X| : X \subseteq E(G), \omega(G - X) \geq c\}$ . For a graph  $G$  and for an integer  $c = 1, 2, \dots, |V(G)| - 1$ , define,

$$\tau_c(G) = \min_{X \subseteq E(G) \text{ and } \omega(G-X) > c} \frac{|X|}{\omega(G - X) - c},$$

where the minimum is taken over all subsets  $X$  of  $E(G)$  such that  $\omega(G - X) - c > 0$ . In this paper, we establish a relationship between  $\lambda_c(G)$  and  $\tau_{c-1}(G)$ , which gives a characterization of the edge-connectivity of a graph  $G$  in terms of the spanning tree packing number of subgraphs of  $G$ . The digraph analogue is also obtained. The main results are applied to show that if a graph  $G$  is  $s$ -hamiltonian, then  $L(G)$  is also  $s$ -hamiltonian, and that if a graph  $G$  is  $s$ -hamiltonian-connected, then  $L(G)$  is also  $s$ -hamiltonian-connected.

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## 1. Introduction

Graphs considered in this paper are finite graphs. Undefined notation and terminology will follow those in [1]. Let  $G$  be a graph. As in [1],  $\kappa'(G)$  and  $\omega(G)$  denote the edge-connectivity and the number of components of  $G$ , respectively. The *spanning tree packing number* of  $G$ , denoted  $\tau(G)$ , is the maximum number of edge-disjoint spanning trees of  $G$ .

Over ten years ago, Catlin left an unpublished note [2] proving a theorem which characterizes the edge-connectivity of a connected graph  $G$  in terms of the spanning tree packing numbers of its subgraphs.

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**Theorem 1.1** (Catlin, [2]). Let  $G$  be a connected graph and let  $k \geq 1$  be an integer. Each of the following holds.

- (i)  $\kappa'(G) \geq 2k$  if and only if  $\forall X' \subseteq E(G)$  with  $|X'| \leq k$ ,  $\tau(G - X') \geq k$ .
- (ii)  $\kappa'(G) \geq 2k + 1$  if and only if  $\forall X' \subseteq E(G)$  with  $|X'| \leq k + 1$ ,  $\tau(G - X') \geq k$ .

Using the fact that for a graph  $G$  and an integer  $m > 0$ ,  $\kappa'(G) \geq m + 1$  if and only if  $\forall e \in E(G)$ ,  $\kappa'(G - e) \geq m$ , one can easily see that Theorem 1.1(ii) follows from Theorem 1.1(i). This theorem of Catlin has been very useful, which motivates the other two authors to seek possible extensions of Theorem 1.1.

The following generalizations of the edge-connectivity and the spanning tree packing number of a graph have been considered in the literature. In [5], Chen et al. defined the *higher order of edge-connectivity* as follows.

Given a graph  $G$ , for an integer  $c \in \{2, \dots, |V(G)|\}$ , define

$$\lambda_c(G) = \min\{|X| : X \subseteq E(G), \omega(G - X) \geq c\}.$$

For  $c \geq 1$ , a  $c$ -forest of  $G$  is a spanning forest of  $G$  with exactly  $c$  components. For a graph  $G$  and for an integer  $c = 1, 2, \dots, |V(G)| - 1$ , the *higher order of edge-toughness* of  $G$ , is defined as

$$\tau_c(G) = \min_{X \subseteq E(G) \text{ and } \omega(G-X) > c} \frac{|X|}{\omega(G - X) - c},$$

where the minimum is taken over all subsets  $X$  of  $E(G)$  such that  $\omega(G - X) - c > 0$ .

Chen, Koh and Peng [5] first determined the relationship between  $\tau_c(G)$  and the maximum number of edge-disjoint  $c$ -forests of  $G$ . When  $c = 1$ , Theorem 1.2 below has been obtained by Nash-Williams [9]. In [6], Chen and Lai presented a short proof of Theorem 1.2 by showing that it is a matroid truncation version of an earlier result in [4].

**Theorem 1.2** (Nash-Williams [9], Chen, Koh and Peng [5], Chen and Lai [6]). For integers  $k \geq 0$  and  $c = 1, 2, \dots, |V(G)| - 1$ ,  $G$  has  $k$  edge-disjoint  $c$ -forests if and only if  $\tau_c(G) \geq k$ .

As  $\lambda_2(G) = \kappa'(G)$  and  $\tau_1(G) = \tau(G)$ ,  $\lambda_c(G)$  and  $\tau_c(G)$  are natural generalizations of  $\kappa'(G)$  and  $\tau(G)$ . In this paper, we shall extend Theorem 1.1 to show that a similar relation exists between  $\lambda_c(G)$  and  $\tau_{c-1}(G)$ . As indicated earlier Theorem 1.1(ii) follows immediately from Theorem 1.1(i), it suffices to generalize Theorem 1.1(i).

**Theorem 1.3.** Let  $G$  be a connected graph and let  $k \geq 1$  and  $c = 2, \dots, |V(G)|$  be integers. Then the following statements are equivalent.

- (i)  $\lambda_c(G) \geq 2(c-1)^2k$ .
- (ii)  $\forall X' \subseteq E(G)$  with  $|X'| \leq (2c^2 - 4c + 1)k$ ,  $G - X'$  has  $k$  edge-disjoint  $(c-1)$ -forests.
- (iii)  $\forall X' \subseteq E(G)$  with  $|X'| \leq (2c^2 - 4c + 1)k$ ,  $\tau_{c-1}(G - X') \geq k$ .
- (iv)  $\forall X, X' \subseteq E(G)$  with  $|X'| \leq (2c^2 - 4c + 1)k$  and with  $\omega(G - (X' \cup X)) \geq c$ ,

$$\frac{|X|}{\omega(G - (X' \cup X)) - (c-1)} \geq k.$$

Let  $D = (V, A)$  be a digraph. The elements of  $A$  are called *arcs*. We use  $e = (u, v)$  for an arc with tail  $u$  and head  $v$  (that is,  $e$  is oriented from  $u$  to  $v$ ). For  $S \subseteq V$ , let  $\Delta^+(S)$  denote the set of arcs with tail in  $S$  and head in  $V - S$ . We use the notation  $\Delta^-(S) = \Delta^+(V - S)$ ,  $\delta^+(S) = |\Delta^+(S)|$ ,  $\delta^-(S) = |\Delta^-(S)|$ .

A digraph  $D$  is called an *arborescence* if  $D$  arises from a tree by orienting the edges in such a way that every vertex but one has one entering arc. A digraph  $D = (V, A)$  is *strongly connected* if there is a directed path from every vertex to any other; *k-arc-connected* if deleting any subset of arcs of less than  $k$  elements leaves a strongly connected digraph. It is well known that the condition that  $D = (V, A)$  is  $k$ -arc-connected is equivalent to the condition that  $\delta^+(X) \geq k$  for any  $\emptyset \neq X \subset V$ .

The following theorem characterizes the arc-connectivity of a digraph in terms of the number of arc-disjoint arborescences of its subgraphs.

**Theorem 1.4.** A digraph  $D = (V, A)$  is  $k$ -arc-connected if and only if  $\forall Y \subseteq A(D)$  with  $|Y| \leq k$ ,  $D - Y$  has  $k$  arc-disjoint spanning arborescences.

It is straightforward to see that when  $c = 2$ , Theorem 1.3 gives Theorem 1.1(i). By the remark after Theorem 1.1, Theorem 1.1 becomes a special case of Theorem 1.3. In fact, Theorem 1.4 also implies Theorem 1.1, as shown in Section 4.

The proofs of Theorems 1.3 and 1.4 will be given in Sections 2 and 3, respectively. In Section 4, we present some of the applications of the main results.

## 2. Proof of Theorem 1.3

By Theorem 1.2, Theorem 1.3(ii) and Theorem 1.3(iii) are equivalent. The equivalence between Theorem 1.3(iii) and Theorem 1.3(iv) follows from the definition of higher order of edge-toughness, as Theorem 1.3(iv) is a restatement of Theorem 1.3(iii). Thus it suffices to show that Theorem 1.3(i) and Theorem 1.3(iv) are equivalent.

We shall first show that Theorem 1.3(i) implies Theorem 1.3(iv). For disjoint subsets  $X, Y \subset V(G)$ ,  $[X, Y]$  denotes the set of edges in  $G$  with one end in  $X$  and the other in  $Y$ , and  $\partial(X) = [X, V(G) - X]$ . If  $H$  is a subgraph of  $G$ , then  $\partial(H) = \partial(V(H))$ .

Let  $Y \subseteq E(G)$  with  $\omega = \omega(G - Y) \geq c \geq 1$ . Suppose that  $W_1, W_2, \dots, W_\omega$  are the components of  $G - Y$ . For each  $(c - 1)$ -subset  $\{W_{i_1}, W_{i_2}, \dots, W_{i_{c-1}}\} \subseteq \{W_1, W_2, \dots, W_\omega\}$ , let

$$X_{i_1, i_2, \dots, i_{c-1}} = \bigcup_{j=1}^{c-1} \partial(W_{i_j}).$$

Since  $G - X_{i_1, i_2, \dots, i_{c-1}}$  has at least  $c$  components, we have, for each  $(c - 1)$ -subset  $\{i_1, i_2, \dots, i_{c-1}\}$  of  $\{1, 2, \dots, \omega\}$ ,

$$|X_{i_1, i_2, \dots, i_{c-1}}| \geq \lambda_c(G). \quad (1)$$

Since we have  $\binom{\omega}{c-1}$  ways to choose these  $c - 1$  components, it follows by (1) that

$$\sum_{\{i_1, i_2, \dots, i_{c-1}\} \subseteq \{1, 2, \dots, \omega\}} |X_{i_1, i_2, \dots, i_{c-1}}| \geq \lambda_c(G) \binom{\omega}{c-1}. \quad (2)$$

For each edge  $e_{ij} \in [W_i, W_j] \subseteq Y$  with  $i \neq j$ ,  $e_{ij} \in X_{i_1, i_2, \dots, i_{c-1}}$  if and only if at least one member of  $\{W_i, W_j\}$  is in  $\{W_{i_1}, W_{i_2}, \dots, W_{i_{c-1}}\}$ . By the Inclusion–Exclusion Theorem,  $e_{ij}$  occurs in exactly  $2 \binom{\omega-1}{c-2} - \binom{\omega-2}{c-3}$  of the  $X_{i_1, i_2, \dots, i_{c-1}}$ 's, and so by (2),

$$\left[ 2 \binom{\omega-1}{c-2} - \binom{\omega-2}{c-3} \right] \cdot |Y| = \sum_{\{i_1, i_2, \dots, i_{c-1}\} \subseteq \{1, 2, \dots, \omega\}} |X_{i_1, i_2, \dots, i_{c-1}}| \geq \lambda_c(G) \binom{\omega}{c-1}. \quad (3)$$

Now we assume  $X, X' \subseteq E(G)$  with  $|X'| \leq (2c^2 - 4c + 1)k$ ,  $Y = X \cup X'$  and  $X \cap X' = \emptyset$ . By (3) and by  $|X'| \leq (2c^2 - 4c + 1)k$ , we have

$$\left[ 2 \binom{\omega-1}{c-2} - \binom{\omega-2}{c-3} \right] \left[ |X| + (2c^2 - 4c + 1)k \right] \geq \lambda_c(G) \binom{\omega}{c-1}. \quad (4)$$

Since  $\lambda_c(G) \geq 2(c - 1)^2 k$ , (4) gives

$$|X| + (2c^2 - 4c + 1)k \geq 2(c - 1)^2 k \frac{\binom{\omega}{c-1}}{2 \binom{\omega-1}{c-2} - \binom{\omega-2}{c-3}}. \quad (5)$$

Straightforward algebraic manipulation yields

$$\frac{\frac{\omega!}{(c-1)!(\omega-c+1)!}}{\frac{2(\omega-1)!}{(c-2)!(\omega-c+1)!} - \frac{(\omega-2)!}{(c-3)!(\omega-c+1)!}} = \frac{\frac{\omega(\omega-1)(\omega-2)!}{(c-1)(c-2)(c-3)!}}{\frac{2(\omega-1)(\omega-2)!}{(c-2)(c-3)!} - \frac{(\omega-2)!}{(c-3)!}} = \frac{\omega(\omega-1)}{(2\omega-c)(c-1)},$$

and so

$$\frac{\binom{\omega}{c-1}}{2\binom{\omega-1}{c-2} - \binom{\omega-2}{c-3}} = \frac{\omega(\omega-1)}{(2\omega-c)(c-1)}. \quad (6)$$

Combine (5) and (6) to get

$$|X| + (2c^2 - 4c + 1)k \geq 2(c-1)^2k \frac{\omega(\omega-1)}{(2\omega-c)(c-1)} = 2(c-1)k \frac{\omega(\omega-1)}{2\omega-c}. \quad (7)$$

Hence

$$\begin{aligned} |X| &\geq 2(c-1)k \frac{\omega(\omega-1)}{2\omega-c} - (2c^2 - 4c + 1)k \\ &= k \left[ \frac{2(c-1)\omega(\omega-1)}{2\omega-c} - \frac{(2c^2 - 4c + 1)(2\omega-c)}{2\omega-c} \right] \\ &= k \left[ \frac{2(c-1)\omega^2 - 2(c-1)\omega - (4c^2 - 8c + 2)\omega + (2c^3 - 4c^2 + c)}{2\omega-c} \right] \\ &= k \frac{2(c-1)\omega^2 - 2(c-1)\omega - (4c^2 - 8c + 2)\omega + (2c^3 - 4c^2 + c)}{2\omega-c}. \end{aligned}$$

It follows that

$$\frac{|X|}{\omega-c+1} \geq k \frac{2(c-1)\omega^2 - 2(c-1)\omega - (4c^2 - 8c + 2)\omega + (2c^3 - 4c^2 + c)}{(2\omega-c)(\omega-c+1)}. \quad (8)$$

By (8) and by Theorem 1.2, it suffices to show that

$$\frac{2(c-1)\omega^2 - 2(c-1)\omega - (4c^2 - 8c + 2)\omega + (2c^3 - 4c^2 + c)}{(2\omega-c)(\omega-c+1)} \geq 1. \quad (9)$$

By algebraic manipulations, the inequality

$$2(c-1)\omega^2 - 2(c-1)\omega - (4c^2 - 8c + 2)\omega + (2c^3 - 4c^2 + c) \geq (2\omega-c)(\omega-c+1) \quad (10)$$

can be expressed as

$$(2c-2)\omega^2 - 2(c-1)\omega - (4c^2 - 8c + 2)\omega + (2c^3 - 4c^2 + c) \geq 2\omega^2 - c\omega - 2(c-1)\omega + c(c-1).$$

By collecting like terms and by removing the common factor  $(c-2)$ , we can rewrite (10) as follows.

$$\begin{aligned} 2(c-2)\omega^2 - (c-2)(4c-1)\omega + (c-2)(2c-1)c &= 2(c-2)\omega^2 - (4c^2 - 9c + 2)\omega + (2c^3 - 5c^2 + 2c) \\ &\geq 0. \end{aligned}$$

Hence (10) is equivalent to

$$(c-2)(\omega-c)(2\omega-2c+1) \geq 0. \quad (11)$$

It follows that when  $\omega \geq c \geq 2$ , Inequality (11) holds, which implies Inequality (9). This proves that Theorem 1.3(i) implies Theorem 1.3(iv).

Now we assume the truth of Theorem 1.3(iv) to prove Theorem 1.3(i). We argue by contradiction and assume that there exists an edge subset  $Y \in E(G)$  with  $|Y| < 2(c-1)^2k$  such that  $\omega(G-Y) \geq c$ . Let  $X, X' \subseteq E(G)$  be subsets of  $Y$  such that  $Y = X \cup X'$ ,  $X \cap X' = \emptyset$ , and such that

$$|X'| = \begin{cases} (2c^2 - 4c + 1)k & \text{if } |Y| \geq (2c^2 - 4c + 1)k \\ |Y| & \text{if } |Y| < (2c^2 - 4c + 1)k. \end{cases}$$

Then  $|X| < k$ . But as  $\omega(G - X' - X) - (c - 1) \geq 1$ ,  $|X| \geq \frac{|X|}{\omega(G - X' - X) - (c - 1)} \geq k$ , a contradiction.

### 3. Proof of Theorem 1.4

Throughout this section, for a digraph  $D$ ,  $V(D)$  and  $A(D)$  denote the vertex set and the arc set of  $D$ , respectively, and  $\omega(D)$  represents the connected components of the undirected graph obtained from  $D$  by converting all directed edges to undirected edges. We need a theorem by Edmonds in our proof.

**Theorem 3.1** (Edmonds, [7]). *A nontrivial digraph  $D$  has  $k$ -arc-disjoint spanning arborescences if and only if for every family of disjoint nonempty sets  $X_1, X_2, \dots, X_t \subseteq V(D)$ ,*

$$\sum_{i=1}^t \delta^+(X_i) \geq k(t - 1),$$

if and only if

$$\min_{Y \subseteq A(G)} \frac{|Y|}{\omega(D - Y) - 1} \geq k, \quad (12)$$

where the minimum runs over all subsets  $Y \subseteq A(G)$  such that  $\omega(D - Y) \geq 2$ .

**Proof of Theorem 1.4.** ( $\implies$ ) Suppose that  $X \subseteq A(G)$  satisfies  $\omega(D - X) \geq 2$ . Count the minimum number  $k$  (say) of outgoing arcs from components of  $D - X$ . Let  $W_1, W_2, \dots, W_\omega$  be the  $\omega(D - X)$  components. Each component  $W_i$  of  $D - X$  is incident with at least  $\delta^+(W_i) \geq k$  edges of  $X$ .

$$|X| = \sum_{i=1}^{\omega} \delta^+(W_i) \geq k\omega(D - X). \quad (13)$$

Since (13) holds for all edge sets of the form  $X = Y \cup Y'$ , where

$$Y \subseteq A(G), |Y| \leq k \quad \text{and} \quad Y \cap Y' = \emptyset,$$

it follows that

$$|Y'| + k \geq |Y \cup Y'| = |X| \geq k\omega(D - X) = k\omega((D - Y) - Y'),$$

and so

$$|Y'| \geq k[\omega((D - Y) - Y') - 1]. \quad (14)$$

Since  $X \subseteq A(D)$  is arbitrary with  $Y \subseteq X$  and  $Y'$  runs over all subsets of  $A(D - Y)$  for all  $Y \subseteq A(D)$  and  $|Y| \leq k$ , where  $\omega((D - Y) - Y') \geq 2$ , it follows by (14) that,

$$\min_{Y' \subseteq A(D - Y)} \frac{|Y'|}{\omega((D - Y) - Y') - 1} \geq k,$$

for all  $Y \subseteq A(D)$  and  $|Y| \leq k$ . Thus (12) must hold with  $D$  replaced by  $D - Y$ , and so by Theorem 3.1,  $D - Y$  has  $k$  arc-disjoint spanning arborescences.

( $\impliedby$ ) We argue by contradiction. Suppose that  $\forall Y \subseteq A(D)$  with  $|Y| \leq k$ ,  $D - Y$  has  $k$  arc-disjoint spanning arborescences but there exists a set  $V_0$  with  $\emptyset \neq V_0 \subset V(D)$  such that  $\delta^+(V_0) < k$ . Let  $Y = [V_0, V(D) - V_0]$ . Then  $\omega(D - Y) \geq 2$  and  $Y = \Delta^+(V_0) \cup \Delta^-(V_0)$  with  $\Delta^+(V_0) \cap \Delta^-(V_0) = \emptyset$ .

Therefore,

$$\frac{|\Delta^+(V_0)|}{\omega(G - \Delta^-(V_0) - \Delta^+(V_0)) - 1} \leq |\Delta^+(V_0)| < k,$$

contrary to (12).  $\square$

#### 4. Applications

Let  $k \geq 0$  be an integer,  $G$  be a graph and  $D(G)$  be an orientation of  $G$ . If  $D(G)$  as a digraph is  $k$ -arc-connected, then  $D(G)$  is a  $k$ -arc-connected orientation of  $G$ . Nash-Williams has characterized graphs that have a  $k$ -arc-connected orientation.

**Theorem 4.1** (Nash-Williams, [9]). *A graph  $G$  has a  $k$ -arc-connected orientation if and only if  $G$  is  $2k$ -edge-connected.*

The following two corollaries of Theorems 1.3 and 1.4 show that Theorem 1.1 also follows from Theorem 1.4.

**Corollary 4.2.** *Let  $G$  be a graph and let  $k > 0$  be an integer. The following are equivalent.*

- (i)  $\kappa'(G) \geq 2k$ .
- (ii)  $G$  has a  $k$ -arc-connected orientation.
- (iii)  $G$  has an orientation  $D = (V, A) = D(G)$  such that  $\forall X \subseteq A(D)$  with  $|X| \leq k$ ,  $D - X$  has  $k$  arc-disjoint spanning arborescence.
- (iv)  $\forall X \subseteq E(G)$  with  $|X| \leq k$ ,  $G - X$  has  $k$  edge-disjoint spanning trees.

**Proof.** By Theorem 4.1, (i)  $\Leftrightarrow$  (ii). By Theorem 1.4(ii)  $\Leftrightarrow$  (iii). Since every spanning arborescence in an orientation of  $G$  is a spanning tree of  $G$ , thus (iii) implies (iv). We assume (iv) to prove (iii). Suppose that  $G$  has an edge cut  $W$  such that  $|W| < 2k$ . Let  $X \subseteq W$  be such that  $|X| = \min\{k, |W|\} \leq k$ . Then  $W - X$  is an edge cut of  $G - X$  with  $|W - X| \leq k - 1$  and so  $G - X$  cannot have  $k$  edge-disjoint spanning trees, contrary to (iv).  $\square$

Notice that  $\kappa'(G) \geq 2k + 1$  if and only if  $\forall e \in E(G)$ ,  $\kappa'(G - e) \geq 2k$ . So Corollary 4.3 below follows directly from Corollary 4.2.

**Corollary 4.3.** *Let  $G$  be a graph and let  $k > 0$  be an integer. The following are equivalent.*

- (i)  $\kappa'(G) \geq 2k + 1$ .
- (ii)  $\forall e \in E(G)$ ,  $\kappa'(G - e) \geq 2k$ .
- (iii)  $\forall e \in E(G)$ ,  $G - e$  has a  $k$ -arc-connected orientation.
- (iv)  $\forall e \in E(G)$ ,  $G - e$  has an orientation  $D(G - e)$  such that  $\forall X \subseteq A(D(G - e))$  with  $|X| \leq k$ ,  $D(G - e) - X$  has  $k$  arc-disjoint spanning arborescences.
- (v)  $\forall X \subseteq E(G)$  with  $|X| \leq k + 1$ ,  $G - X$  has  $k$  edge-disjoint spanning trees.

A graph  $G$  is *hamiltonian-connected* if  $\forall u, v \in V(G)$ ,  $G$  has a spanning  $(u, v)$ -path. The *line graph* of a graph  $G$ , denoted by  $L(G)$ , has  $E(G)$  as its vertex set, where two vertices in  $L(G)$  are adjacent if and only if the corresponding edges in  $G$  are adjacent. In 1986, Zhan proved:

**Theorem 4.4** (Zhan, [11]). *If  $G$  is a 4-edge-connected graph, then the line graph  $L(G)$  is hamiltonian-connected.*

For integers  $n \geq 3$  and  $s \geq 0$  with  $s \leq n - 3$ , a graph  $G$  of order  $n$  is *s-hamiltonian*, if the removal of any  $k$  vertices,  $0 \leq k \leq s$ , results in a hamiltonian graph. Lesniak-Foster proved a nice result.

**Theorem 4.5** (Lesniak-Foster, Lemma 3 in [8]). *If  $G$  is 1-hamiltonian, then  $L(G)$  is 1-hamiltonian.*

We can easily extend Theorems 4.4 and 4.5 respectively by using Corollary 4.2 and the following Theorems 4.6 and 4.7.

For a graph  $G$ , let  $O(G) = \{v \in V(G) : d_G(v) \text{ is odd}\}$ . A connected graph  $G$  is *eulerian* if  $O(G) = \emptyset$ . A spanning closed trail of  $G$  is also referred as a *spanning eulerian subgraph* of  $G$ . A subgraph  $H$  of  $G$  is *dominating* if  $G - V(H)$  is edgeless. (Note the difference between a dominating vertex subset and a dominating subgraph.) If a closed trail  $C$  of  $G$  satisfies  $E(G - V(C)) = \emptyset$ , then  $C$  is a *dominating eulerian subgraph*. A well known relationship between dominating eulerian subgraphs in  $G$  and hamiltonian cycles in  $L(G)$  is given by Harary and Nash-Williams.

**Theorem 4.6** (Harary and Nash-Williams, [10]). *Let  $G$  be a connected graph with at least 3 edges. The line graph  $L(G)$  is hamiltonian if and only if  $G$  has a dominating eulerian subgraph.*

If  $P = v_0v_1 \cdots v_k$  denotes a trail (or path, respectively), of  $G$ , and if  $E(P) = \{e_1, e_2, \dots, e_k\}$  is an edge set such that for  $i = 1, 2, \dots, k$ ,  $e_i = v_{i-1}v_i$ , then  $P$  is called a  $(v_0, v_k)$ -trail (or path, respectively) of  $G$ , and an  $(e_1, e_k)$ -trail (or path, respectively) of  $G$ . The vertices  $v_1, v_2, \dots, v_{k-1}$  are called the *internal vertices* of  $P$ . The edges  $e_1, e_k$  are called the *end edges* of  $P$ . A trail  $P$  of  $G$  is dominating if every edge of  $G$  is incident with an internal vertex of  $P$ . With similar arguments in [10] (see page 74 of [10]), the following can be similarly proved.

**Theorem 4.7.** *Let  $G$  be a connected graph with  $|E(G)| \geq 3$ . Then  $L(G)$  is hamiltonian-connected if and only if for any pair of edges  $e_1, e_2 \in E(G)$ ,  $G$  has a dominating  $(e_1, e_2)$ -trail.*

**Theorem 4.8** (Catlin and Lai, [3]). *Let  $G$  be a graph and  $e_1, e_2 \in E(G)$ . If  $G$  has two edge-disjoint spanning trees, then either  $G$  has a spanning  $(e_1, e_2)$ -trail, or  $\{e_1, e_2\}$  is an edge cut of  $G$  such that both components of  $G - \{e_1, e_2\}$  contain at least one edge.*

Let  $n \geq 3$  and  $s \geq 0$  be integers with  $0 \leq s \leq n - 3$ . A graph  $G$  of order  $n$  is  $s$ -hamiltonian-connected, if the removal of any  $k$  vertices,  $0 \leq k \leq s$ , results in a hamiltonian-connected graph.

**Corollary 4.9.** *Let  $s \geq 2$  be an integer. Each of the following holds.*

- (i) *If a graph  $G$  is  $(s + 2)$ -edge-connected, then  $L(G)$  is  $s$ -hamiltonian.*
- (ii) *If a graph  $G$  is  $s$ -hamiltonian, then  $L(G)$  is also  $s$ -hamiltonian.*

**Proof.** (i) Let  $S \in V(L(G))$  with  $|S| = s$  and  $S'$  be the corresponding edge set in  $G$ . Let  $S'' \subset S'$  with  $|S''| = s - 2 \geq 0$ . Since  $G$  is  $(s + 2)$ -edge-connected,  $G - S''$  is 4-edge-connected. By Corollary 4.2 (i) and (iv),  $G - S'$  has two edge-disjoint spanning trees, and so  $G - S'$  has a spanning eulerian subgraph. By Theorem 4.6,  $L(G - S)$  is hamiltonian. Thus  $L(G)$  is  $s$ -hamiltonian.

(ii) If  $G$  is  $s$ -hamiltonian, then  $\kappa'(G) \geq \kappa(G) \geq s + 2$ . Thus by (i),  $L(G)$  must also be  $s$ -hamiltonian.  $\square$

**Corollary 4.10.** *Let  $s \geq 0$  be an integer.*

- (i) *If a graph  $G$  is  $(s + 4)$ -edge-connected, then  $L(G)$  is  $s$ -hamiltonian-connected.*
- (ii) *If a graph  $G$  is  $s$ -hamiltonian-connected, then  $L(G)$  is also  $s$ -hamiltonian-connected.*

**Proof.** (i) Let  $S \in V(L(G))$  with  $|S| = s$  and  $S'$  be the corresponding edge set in  $G$ . Since  $G$  is  $(s + 4)$ -edge-connected,  $G - S'$  is 4-edge-connected. By Corollary 4.2(i) and (iv),  $G - S'$  has two edge-disjoint spanning trees. Since  $G - S'$  has no 2-edge-cut  $X$  such that both components of  $G - X$  contain at least one edge, by Theorem 4.8, for any  $e_1, e_2 \in E(G - S')$ ,  $G - S'$  has a spanning  $(e_1, e_2)$ -trail. By Theorem 4.7,  $L(G - S)$  is hamiltonian-connected. Thus  $L(G)$  is  $s$ -hamiltonian-connected.

(ii) When  $s = 0$ , by Theorem 4.7, it suffices to show that for any edges  $e_1, e_2 \in E(G)$ ,  $G$  has dominating  $(e_1, e_2)$ -trail.

We assume that  $e_1 = u_1v_1$  and  $e_2 = u_2v_2$ . Since  $G$  is hamiltonian-connected,  $G$  has a hamiltonian  $(u_1, u_2)$ -path  $P$ . If  $e_1, e_2 \in E(P)$ , then since  $P$  is a  $(u_1, u_2)$ -path,  $e_1$  and  $e_2$  must be the two end edges of the path and so  $P$  is a dominating  $(e_1, e_2)$ -trail of  $G$ . If  $e_1, e_2 \notin E(P)$ , then  $G[E(P) \cup \{e_1, e_2\}]$  is a dominating  $(e_1, e_2)$ -trail of  $G$ . If  $|E(P) \cap \{e_1, e_2\}| = 1$ , then we may assume that  $e_1 \in E(P)$  and  $e_2 \notin E(P)$ . Since  $P$  is a  $(u_1, u_2)$ -path,  $e_1$  must be an end edge of  $P$ , and so  $G[E(P) \cup \{e_2\}]$  is a dominating  $(e_1, e_2)$ -trail of  $G$ .

We assume that  $s \geq 1$ . Since  $G$  is  $s$ -hamiltonian-connected, then  $\kappa'(G) \geq \kappa(G) \geq s + 3$ . Let  $S \in V(L(G))$  with  $|S| = s$  and  $S'$  be the corresponding edge set in  $G$ . Let  $S'' \subset S'$  with  $|S''| = s - 1 \geq 0$ . Since  $G$  is  $(s + 3)$ -edge-connected,  $G - S''$  is 4-edge-connected. By Corollary 4.2(i) and (iv),  $G - S'$  has two edge-disjoint spanning trees. Since  $\kappa'(G - S') \geq 3$ ,  $G - S'$  has no 2-edge-cut  $X$  such that both components of  $G - X$  contain at least one edge. By Theorem 4.8, for any  $e_1, e_2 \in E(G - S')$ ,  $G - S'$  has a spanning  $(e_1, e_2)$ -trail. By Theorem 4.7,  $L(G - S)$  is hamiltonian-connected. Thus  $L(G)$  is  $s$ -hamiltonian-connected.  $\square$

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